Problem Set 1 - Analysis [1]

Due date: 08/26

Question 1 Suppose P, Q, and R are statements. Use the truth table to show that the following statements are always true.

(1) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ (*modus ponens*)

(2) $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$ (*modus tollens*)

(3) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ (*syllogism*)

Question 2 Suppose A and B are sets. Show the following statements are equivalent.

(1) $A \subset B$.

(2) $A \cup B = B$.

(3) $A \cap B = A$.

Question 3 In class, we defined the uniqueness existential quantifier: $\exists!$. For example, "$\exists! x \in X$ such that..." means "there exists a unique x in X such that..." However, such statements can be defined using $\forall$, $\exists$, and logical operators.
Write a symbolic statement that is equivalent to "$\exists! x \in X$ such that P(X)" without using !.

Question 4 Let us consider a function $f : \mathbb{R}^n \to \mathbb{R}$.

(a) If f is strictly increasing, is it strongly increasing?

(b) If f is strongly increasing, is it strictly increasing?

Question 5 Use the epsilon-delta definition of a limit to prove that

$$\lim_{x \to 1} \frac{x}{x^2+1} = \frac{1}{2}$$

Question 6 Use the Squeeze Theorem, prove that

$$\lim_{x \to \infty} (1 + \frac{1}{x^2})^x = 1$$

(Hint: $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ and $\lim_{x\to\infty} e^{\frac{1}{x}} = 1$)

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Optional Questions

Question 7 Prove the generalized De Morgan's Law: for sets A_1, A_2, $...A_n$,

(a) $\left(\bigcup_{i=1}^n A_i\right)^C = \bigcap_{i=1}^n A_i^C$

(b) $\left(\bigcap_{i=1}^n A_i\right)^C = \bigcup_{i=1}^n A_i^C$

Question 8 Revisit the example from lecture. Let $X = \{a, b, c\}$, and $\succ \subset X \times X$ be a preference relation. Moona has the following preferences:

$$a \succ_M b, \quad a \succ_M c, \quad b \succ_M c$$

Nina has the following preferences:

$$a \succ_N a, \quad a \succ_N b, \quad a \succ_N c, \quad b \succ_N c, \quad b \succ_N b, \quad c \succ_N c$$

Let us consider the following definitions:
Definition. We say a relation R defined on $X \times X$ is *complete* if for any $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$. We say a relation R defined on $X \times X$ is *transitive* if for any $x, y, z \in X$, when $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

(a) Are $\succ_M$ and $\succ_N$ complete?

(b) Are $\succ_M$ and $\succ_N$ transitive?

Question 9 Consider a lexicographical preference on $\mathbb{R}^2$ as follows: $(x, y) \succeq (x', y')$ if $x > x'$ or $x = x'$ and $y \geq y'$.

(a) Characterize $\succeq$ using a set expression.

(b) Show that $\succeq$ is complete and transitive.

Question 10 Let E be a nonempty subset of an ordered set. Suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.