Math Camp 2025, UCSB
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Problem Set 2 - Linear Algebra [1]

Due date: 09/02

This problem set will help you review the key concepts from Week 2. You are asked to submit this assignment using LaTeX or Rmarkdown. The purpose is to give you an opportunity to use LaTeX and R Markdown if you haven't. Please have only one member in your group send an email to both instructors by the due date with other members cc'ed in the email.

1. Let S be a sample space and let $\mathcal{B}$ be a σ-algebra on S. Use the properties of a σ-algebra to prove that:
   (a) $S \in \mathcal{B}$.
   (Hint: start with that $\mathcal{B}$ should be nonempty.)
   (b) $\mathcal{B}$ is closed under countable intersections.

2. Let $\mathbb{P}$ be a probability measure on a sample space S with σ-algebra $\mathcal{B}$, and let $A, B \in \mathcal{B}$. Prove the following properties:
   (a) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
   (b) $\mathbb{P}(A) \leq 1$
   (c) $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(B \cap A)$
   (d) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
   (e) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

3. Let S be a sample space with σ-algebra $\mathcal{B}$, and let $A, B \in \mathcal{B}$. Prove that if A and B are independent, then the following pairs of events are also independent:
   (a) A and B^C
   (b) A^C and B
   (c) A^C and B^C

4. Let $\mathbb{P}$ be a probability measure on a sample space S with σ-algebra $\mathcal{B}$. Let $A, B, C \in \mathcal{B}$.
   (a) Show that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ does not imply that the events A, B, C are pairwise independent. (Hint: you only need to provide a counterexample).
   (b) What additional conditions are needed to guarantee that A, B, and C are mutually independent?

[1] This problem set is adapted from materials prepared by former Math Camp instructors Seonmin Will Heo, Eunseo Kang, and Woongchan Jeon.

5. A test is used to detect the presence of a disease. The test has the following properties:
   - If a patient has the disease, the test always returns a positive result.
   - If a patient does not have the disease, the test returns a false positive with probability 0.005.

   Suppose the probability of having the disease is 0.001.
   If a patient receives a positive test result, what is the probability that they have the disease?

6. A variable X is lognormally distributed if $Y = \ln(X)$ is normally distributed with mean μ and variance σ^2. That is, $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$. Let the transformation be defined by $x = g(y) = e^y$ so that $y = g^{-1}(x) = \ln(x)$.
   (a) Derive $f_X(x)$.
   (b) Derive $\mathbb{E}[X^t]$ using $M_Y(t)$. What are $\mathbb{E}[X]$ and $V(X)$?

7. Let us consider the Law of Iterated Expectations in the continuous case. Suppose that $\mathbb{E}[Y] < \infty$. Prove the following results:
   (a) $\mathbb{E}[Y] = \mathbb{E}\Big[\mathbb{E}[Y|X]\Big]$
   (b) $\mathbb{E}[Y|X] = \mathbb{E}\Big[\mathbb{E}[Y|X,Z] \mid X\Big]$

8. Assume there are n volunteers elegible to receive a treatment. For each unit $i \in \{1, \cdots, n\}$, define the treatment indicator

$$D_i = \begin{cases} 1 & \text{if unit } i \text{ is treated} \\ 0 & \text{otherwise} \end{cases}$$

   Let $(D_1, \cdots, D_n)$ be the vector of the treatment indicators of all units. Due to capacity constraints, only $n_1 (< n)$ units can be treated: $\sum_{i=1}^n D_i = n_1$.
   (a) What is the total number of distinct treatment assignment vectors $(D_1, \cdots, D_n)$ we can construct?

   We say that treatment is randomly assigned if $(D_1, \cdots, D_n)$ are random variables, and if for any vector of n numbers $(d_1, \cdots, d_n) \in \{0,1\}^n$ such that $\sum_{i=1}^n d_i = n_1$,

$$P(D_1 = d_1, \cdots, D_n = d_n) = \frac{1}{\binom{n}{n_1}}$$

   That is, random assignment generates uniform treatment probabilities across units. Assuming the treatment is randomly assigned, answer the following:
   (b) For any unit $i \in \{1, \cdots, n\}$, what is $P(D_i = 1)$?
   (c) For any units $i \neq j$, what is $P\left(D_i = 1 \;\wedge\; D_j = 1\right)$? Is it true that unit i getting treated is independent from unit j getting treated?