Problem Set 2 - Analysis[1]

Question 1 Let x and y be any real numbers. Show that $||x| - |y|| \leq |x - y|$ (this is called the reverse triangle inequality)

Question 2 Find the Taylor polynomial of degree 5 ($n = 5$) for the function $f(x) = e^x$ around the point $x = 0$. Then, evaluate the polynomial at $x = 1$.

Question 3 Let $x(t)$ be differentiable. Show that $\frac{\frac{dx(t)}{dt}}{x} = \frac{d\log(x(t))}{dt}$.

Question 4 Determine whether the following functions are convex/concave/quasi-convex/quasi-concave.
(1) $f(x, y) = \sqrt{x} + \sqrt{y}$
(2) $g(x, y) = \sqrt{xy}$

Question 5
Let (X, d) be a metric space, where $X = \mathbb{R}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, define $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

(1) Consider $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n = 1, 2, ...$ For any n, is I_n an open set or a closed set (or both)? Find $\bigcap_{n=1}^{\infty} I_n$. Is it an open set or a closed set (or both)?

(2) Consider $J_n = \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right], n = 1, 2, ...$ For any n, is I_n an open set or a closed set (or both)? Find $\bigcup_{n=1}^{\infty} J_n$. Is it an open set or a closed set (or both)?

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Question 6
(Comment: You can use Euler's theorem for homogeneous functions here, but only after proving it.)

(1) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree one. Prove that

$$f(K, L) = K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L}$$

(2) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree r. We can also have a similar formula:

$$f(K, L) = g(r)(K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L})$$

where $g(r)$ is some function of r. Find $g(r)$.

Questions 7, 8 and 9 are optional.

Question 7
Prove that $f(K, L) = AK^{\alpha}L^{(1-\alpha)}$ is quasiconcave.

Question 8
Prove that a sequence of vectors in R^m converges if and only if all m sequences of its components converge in $\mathbb{R}$. More precisely, prove that the sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ converges to $\mathbf{x}$ if and only if for every $i = 1, ..., m$, the sequence of real numbers $\{x_{in}\}_{n=1}^{\infty}$ converges to a limit x_i. (Here, $\mathbf{x}_n = (x_{1n}, x_{2n}, ..., x_{mn})$ and $\mathbf{x} = (x_1, x_2, ..., x_m)$.)

Question 9
Consider the following sets. Determine if they are bounded. If the set is bounded, provide an M and a $\mathbf{x}$ such that $B_M(\mathbf{x})$ contains the set.

1. $A = \{x | x \in \mathbb{R} \wedge x^2 \leq 10\}$
2. $B = \left\{x | x \in \mathbb{R} \wedge x + \frac{1}{x} < 5\right\}$
3. $C = \{(x, y) | (x, y) \in \mathbb{R}^2_+ \wedge xy < 1\}$
4. $D = \{(x, y) | (x, y) \in \mathbb{R} \wedge |x| + |y| \leq 10\}$