Math Camp 2025 – Applied Micro

Evaluating Estimators, Convergence and Inference

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Evaluating Estimators

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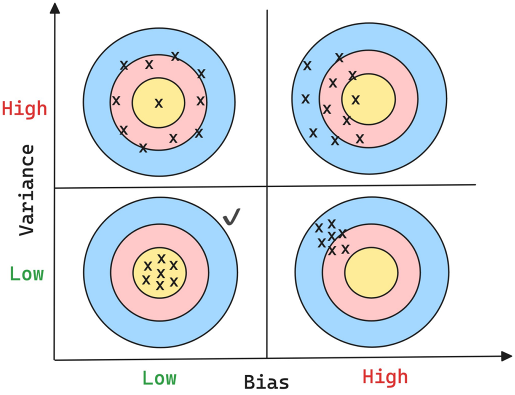

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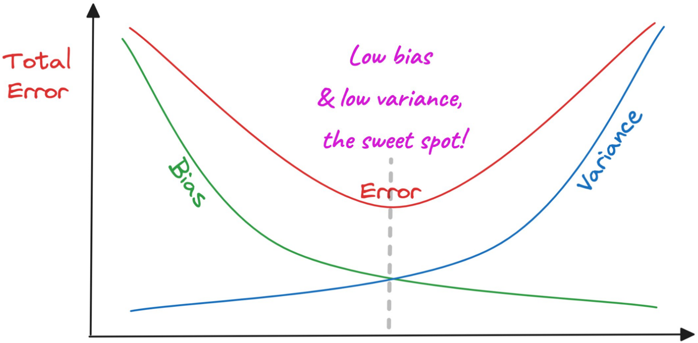

Bias of an estimator

Definition: Bias of an estimator $\hat{\theta}_n$ of θ:

$$\text{Bias}(\hat{\theta}_n) = \mathbb{E}[\hat{\theta}_n] - \theta$$

Example: Variance Estimators

Two estimators for σ^2:

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2, \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

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$$\text{Bias}(S_n^2) = 0, \qquad \text{Bias}(\hat{\sigma}_n^2) = \frac{\sigma^2}{n}$$

$$\text{Var}(S_n^2) = \frac{2\sigma^4}{n-1}, \qquad \text{Var}(\hat{\sigma}_n^2) = \frac{2(n-1)\sigma^4}{n^2}$$

Mean Squared Error (MSE) – Key Facts

Definition:

$$\mathsf{MSE}(\hat{\theta}) = \mathbb{E}\big[(\hat{\theta} - \theta)^2\big]$$

Key Points:

- Measures the *average squared difference* between an estimator $\hat{\theta}$ and the true parameter θ.

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- Lower MSE $\Rightarrow$ estimator is *closer to the true value on average*.
- Penalizes *large errors more heavily* due to squaring.
- Can be decomposed into a formula involving variance and bias.

Mean Squared Error (MSE)

Show that the Mean Squared Error (MSE) can be expressed as:

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Show that the Mean Squared Error (MSE) can be expressed as:

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Recall:

$$\mathsf{Var}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2].$$

$$\mathsf{Bias}(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta.$$

Should we use this to decide between S_n^2 and $\hat{\sigma}_n^2$?

$$\text{Bias}(S_n^2) = 0, \qquad \text{Bias}(\hat{\sigma}_n^2) = \frac{\sigma^2}{n}$$

$$\text{Var}(S_n^2) = \frac{2\sigma^4}{n-1}, \qquad \text{Var}(\hat{\sigma}_n^2) = \frac{2(n-1)\sigma^4}{n^2}$$

Convergence in Probability

Let $\mathbf{U}_1, \mathbf{U}_2, \cdots$ be a sequence of random vectors. This sequence **converges in probability** to a random vector $\mathbf{V}$ if for any $\varepsilon > 0$:

$$\lim_{n \to \infty} P\Big(||\mathbf{U}_n - \mathbf{V}|| < \varepsilon \Big) = 1.$$

Alternatively, we write $\mathbf{U}_n \xrightarrow{p} \mathbf{V}$.

Weak Law of Large Numbers

Let $\{\mathbf{X}_1, \cdots, \mathbf{X}_n\}$ be a random sample and let $\mathbf{X}$ be a random vector with the same probability distribution as $\mathbf{X}_i$'s.

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Assume that $\mathbb{E}[\mathbf{X}] < \infty$. Define $\bar{\mathbf{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$. Then for every $\varepsilon > 0$,

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That is, $\bar{\mathbf{X}}_n$ converges in probability to $\mathbb{E}[\mathbf{X}]$. This is known as the **weak law of large numbers**.

More on Convergence

Suppose $Y_n \xrightarrow{p} Y$ and $Z_n \xrightarrow{p} Z$. Then

1. $cY_n \xrightarrow{p} cY$ where $c \in \mathbb{R}$
2. $Y_n + Z_n \xrightarrow{p} Y + Z$
3. $Y_n Z_n \xrightarrow{p} YZ$

More on Convergence

Let $\{\mathbf{X}_1, \cdots, \mathbf{X}_n\}$ be a random sample. Let $\hat{\theta}_n(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ be an estimator for the parameter θ, based on a sample size n. Then $\hat{\theta}_n$ is a **consistent estimator** for θ if

$$\hat{\theta}_n \xrightarrow{p} \theta$$

Central Limit Theorem

Let $\{\mathbf{X}_1, \cdots, \mathbf{X}_n\}$ be a random sample and let $\mathbf{X}$ be a random vector with the same probability distribution as $\mathbf{X}_i$'s. If $\mathbb{E}|\mathbf{X}\mathbf{X}^T| < \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i - \mathbb{E}[\mathbf{X}]\right) \rightsquigarrow N(\mathbf{0}, \Sigma)$$

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where $\Sigma = \mathbb{E}\Big[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T\Big]$ and $\rightsquigarrow$ is short-hand for "distributed in the limit."

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Note that from our WLLN, $\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i - \mathbb{E}[\mathbf{X}]$ will converge in probability to zero. It converges at rate $\sqrt{n}$, however, so by multiplying by $\sqrt{n}$, we "grow" this value at the same rate it "shrinks," thus ensuring we get a distribution instead of a simply zero.